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An Estimate for Multivariate Interpolation

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Suppose f is a distribution on R^n, all of whose kth order derivatives are in $L^p(R^n)$ and k is large enough to imply that f is continuous, namely, $kp > n$. If the values of f on a grid of points (not necessarily regular) are in l^p, we show that f is in $L^p(R^n)$ and there is an estimate on the L^p norm of f in terms of the l^p norms of these values and the L^p norms of its kth order derivatives. In the case that these values are all zero, this result is useful in obtaining estimates for certain types of multivariate interpolation schemes. An application to generalized splines is given.

1. INTRODUCTION

Suppose f is a continuously differentiable function on the interval $I = [a, b]$ such that $f(t_i) = 0$, $i = 0, 1,..., m$, at the points $a = t_0 < t_1 < \cdots < t_m = b$. If $h = \max_i(t_i - t_{i-1})$, it is not difficult to obtain an estimate on the L^2 norm of f in terms of the L^2 norm of its derivative f'. Indeed, as is well known, an application of elementary calculus results in

$$\|f\| \leqslant C_1 h \|f'\| \tag{1}$$

where $\|f\| = \{\int_a^b |f(t)|^2\, dt\}^{1/2}$ and C_1 is a constant independent of f and h. If f is smoother, then under similar conditions (assuming $k \leqslant m$)

$$\|f\| \leqslant C_k h^k \|f^{(k)}\| \tag{2}$$

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where $f^{(k)}$ denotes the kth derivative of f. Analogous estimates involving more general L^p norms also hold.

Inequalities such as (1) and (2) are very useful in obtaining error estimates for various one dimensional interpolation schemes. For example, if g is a continuously differentiable function on I and s is its piecewise linear spline interpolant on the points mentioned above, then using the well known fact that $\| g' - s' \| \leqslant \| g' \|$ and applying (1) with $f = g - s$, we have the error estimate $\| g - s \| \leqslant C_1 h \| g' \|$.

One of the purposes of this paper is to state and prove a multivariate analogue of (1) and (2) in the L^p norm without special assumptions on the geometry of the set of zeros of f. The result follows from a theorem, given below, concerning a relationship between the pointwise values of f on certain countable subsets of R^n, the L^p norm of f, and the L^p norm of its derivatives. We should also mention that there are many other results relating the values of linear functionals and Sobolev norms; for example, see [3, 4, 6] or [10].

Before introducing various technical details we observe that if the number of variables is n, $n \geqslant 2$, and the set of zeros of f is some discrete lattice in R^n, then an estimate such as (1) involving only the first gradient and L^2 norms over some open set is not possible. The reason can be easily seen from the following argument: If such an estimate were to hold for smooth functions then, since the inequality depends only on the L^2 norm of grad f, it should hold for all distributions whose gradient is in L^2. However, if the number of variables is two or more, such f need not be continuous and need not make sense pointwise; in particular, the hypothesis concerning the vanishing of f on a discrete set is meaningless.

It should be clear that appropriate analogues of (2) in the L^p norm will involve a lower bound on the order of differentiation in terms of p and the number of variables. These bounds should be directly related, of course, to the number of derivatives in L^p that a distribution needs in order to be equivalent to a continuous function.

As is customary, generic constants which occur in various expressions below will be denoted by the symbol C with or without subscripts. These constants usually depend on p and the number of variables and need not be the same at every occurrence. By keeping track of these constants and how they arise, one may obtain a rough estimate on them in terms of all the parameters involved.[1]

[1] We wish to thank the referee for bringing Refs. [9, 10] to our attention and for pointing out that perhaps the methods in [10] may lead to more accurate estimates of these constants.

2. Main Results

We will always be dealing with complex valued functions and distributions defined on some open subset Ω of R^n and use standard multi-index notation (see [1, p. 1]). Unsubscripted absolute value notation denotes either Euclidean length or multi-index length, depending on the context. The symbol $\dot{W}^{k,p}(\Omega)$, with p satisfying $1 \leqslant p \leqslant \infty$ and k a non-negative integer, denotes the space of distributions, f, all of whose kth order derivatives are in $L^p(\Omega)$; this space is equipped with the semi-norm $|f|_{k,p,\Omega} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(\Omega)}$ where $\|\phi\|_{L^p(\Omega)} = (\int_\Omega |\phi(x)|^p\, dx)^{1/p}$ if $1 \leqslant p < \infty$ with the usual modification in the case $p = \infty$. Note that $|f|_{0,p,\Omega} = \|f\|_{L^p(\Omega)}$. $W^{k,p}(\Omega) = \bigcap_{j=0}^k \dot{W}^{j,p}$ and is equipped with the norm $\|f\|_{k,p,\Omega} = \sum_{j=0}^k |f|_{k,p,\Omega}$. When the symbol Ω does not appear in the subscript it is to be understood that $\Omega = R^n$; we do this to simplify the notation. Thus, for example, $\|f\|_{L^p} = \|f\|_{L^p(R^n)}$ and $|f|_{k,p} = |f|_{k,p,R^n}$.

Recall that if f is in $\dot{W}^{k,p}(\Omega)$ then $D^\alpha f$ is locally in L^p for all α which satisfy $0 \leqslant |\alpha| \leqslant k$; more details can be found, for example, in [9]. Furthermore, if $kp > n$ then the equivalence class of measurable functions corresponding to f contains a continuous function; to avoid confusion, when discussing such f we will always assume that we are dealing with a continuous function. The following lemma, which is used in the proof of Theorem 1, gives another local property of such f.

Lemma 1. *Suppose f is in $\dot{W}^{k,p}(\Omega)$, $kp > n$, and Q is a cube with sides of length h whose closure is contained in Ω. Then*

$$\|f\|_{L^p(Q)} \leqslant h^{n/p}\, |f(x_0)| + \sum_{j=1}^{k} C_j h^j\, |f|_{j,p,Q} \tag{3}$$

where x_0 is any point contained in Q and the C_j's are positive constants independent of f, Q, and Ω.

Proof. Suppose f is k times continuously differentiable on the closure of Q and, for any x in Q, write

$$f(x) = f(x_0) + A + B \tag{4}$$

where $A = \sum_{1 \leqslant |\alpha| < k} C_\alpha D^\alpha f(x)(x - x_0)^\alpha$ and $B = \sum_{|\alpha|=k} C_\alpha \int_0^r t^{k-1} D^\alpha f(x_0 + t\omega)\, \omega^\alpha\, dt$ where $\omega = (x - x_0)/r$ and $r = |x - x_0|$. It is easy to see that

$$\|A\|_{L^p(Q)} \leqslant \sum_{j=1}^{k-1} C_j h^j\, |f|_{j,p,Q}. \tag{5}$$

To obtain an estimate on the $L^p(Q)$ norm of B, consider a typical term, apply Holder's inequality using the fact that $kp > n$, and write

$$\left|\int_0^r t^{k-1} D^\alpha f(x_0 + t\omega)\,\omega^\alpha\,dt\right|^p \leqslant C r^{kp-n} \int_0^r |D^\alpha f(x_0 + t\omega)|^p\, t^{n-1}\,dt.$$

Call the term on the left hand side of the above inequality B_α^p. If $\{(r, \omega): 0 \leqslant r \leqslant \rho(\omega),\ \omega \in R^n \text{ with } |\omega| = 1\}$ describes the closure of Q in terms of spherical coordinates centered at x_0, then integrating both sides of the last inequality over Q in terms of these coordinates and interchanging orders of t and r integration results in

$$\begin{aligned}\int_Q B_\alpha^p\,dx &= \int_{|\omega|=1} \int_0^{\rho(\omega)} B_\alpha^p r^{n-1}\,dr\,d\omega \\ &\leqslant C \int_{|\omega|=1} \int_0^{\rho(\omega)} |D^\alpha f(x_0 + t\omega)|^p\, t^{n-1} r^{kp-1}\,dr\,dt\,d\omega \\ &\leqslant C h^{kp} \int_{|\omega|=1} \int_0^{\rho(\omega)} |D^\alpha f(x_0 + t\omega)|^p\, t^{n-1}\,dt\,d\omega\end{aligned}$$

where $d\omega$ denotes the usual rotation invariant Lebesgue measure on the unit sphere. Rewriting the last inequality results in

$$\int_Q B_\alpha^p\,dx \leqslant C h^{kp}\, \|D^\alpha f\|_{L^p(Q)}^p$$

which, of course, implies that

$$\|B\|_{L^p(Q)} \leqslant C h^k\, |f|_{k,p,Q}. \tag{6}$$

Estimates (5) and (6) together with formula (4) imply the desired result for f.

The fact that (3) holds for all f in $\dot{W}^{k,p}(\Omega)$ follows from a standard argument using mollifiers. To wit, let ϕ be an infinitely differentiable function with support in $|x| < 1$ such that $\int_{|x|<1} \phi(x)\,dx = 1$. Write $f_t(x) = \int f(x-y)\,\phi(y/t)\,t^{-n}\,dy$ which is well defined for all x in closure of Q if t is sufficiently small. Furthermore it is easy to check that f_t is k times continuously differentiable in the closure of Q, $\lim_{t\to 0} f_t(x_0) = f(x_0)$, and $\lim_{t\to 0} D^\alpha f_t = D^\alpha f$ in $L^p(Q)$ for $0 \leqslant |\alpha| \leqslant k$. The last remark implies that (3) holds for f_t if t is sufficiently small and thus, in the limit, (3) also holds for f. Thus the proof of the lemma is complete.

THEOREM 1. *Suppose f is in $\dot{W}^{k,p}(R^n)$ and $kp > n$. Let Z be any subset of*

R^n such that $h = \max\{\text{distance}(x, Z): x \in R^n\}$ is finite. If $\sum_{z \in Z} |f(z)|^p$ is finite then f is in $L^p(R^n)$ and

$$\|f\|_{L^p} \leqslant C\left\{\left(h^n \sum_{z \in Z} |f(z)|^p\right)^{1/p} + h^k |f|_{k,p}\right\} \tag{7}$$

where C is a constant independent of f, h, and Z.

Proof. Using the change of variables $x \to hx$ and the homogeneity of the various semi-norms with respect to this change, one sees that it suffices to prove the theorem for the case $h = 1$. Assuming $h = 1$, partition R^n into congruent disjoint cubes with sides of length three and call this partition P. Since each cube in P properly contains a ball of radius one, it follows that each such cube, Q, contains at least one point of Z, call it z_Q. Applying the lemma to f, Q, and z_Q, raising the right hand side of (3) to the pth power while treating the left accordingly, and summing over all cubes in the partition results in, after taking the pth root,

$$\|f\|_{L^p} \leqslant C\left(\sum_{Q \in P} |f(z_Q)|^p\right)^{1/p} + A \tag{8}$$

where $A = C\sum_{j=1}^{k} |f|_{j,p}$. Now recall (for example, see [1]) that for any positive ε

$$|f|_{j,p} \leqslant C\{\varepsilon^{-j} |f|_{0,p} + \varepsilon^{k-j} |f|_{k,p}\}$$

for $j = 1, \ldots, k-1$ and use this to estimate $|f|_{j,p}$ in A to get

$$A \leqslant C_1(\varepsilon)\, \|f\|_{L^p} + C_2(\varepsilon)\, |f|_{k,p}$$

where $C_1(\varepsilon) = C(1 - \varepsilon^{1-k})/(\varepsilon - 1)$. Substitute the last estimate for A into (8), choose ε sufficiently large so that $C_1(\varepsilon)$ is no greater than one-half, and subtract $C_1(\varepsilon)\, \|f\|_{L^p}$ from both sides of the resulting inequality. Finally, dividing both sides of this expression by $1 - C_1(\varepsilon)$ results in (7) with Z replaced by $Z_P = \{z_Q: Q \in P\}$. However, Z_P is a subset of Z so the desired result easily follows from this.

We now list two immediate corollaries, the first one of which is the result discussed in the Introduction.

COROLLARY 1. *Suppose f is in $\dot{W}^{k,p}(R^n)$ and $kp > n$. Let $Z = \{x \in R^n: f(x) = 0\}$ and suppose that $h = \max\{\text{distance}(x, Z): x \in R^n\}$ is finite. Then $D^\alpha f$ is in $L^p(R^n)$ for all α such that $0 \leqslant |\alpha| \leqslant k$ and*

$$|f|_{j,p} \leqslant Ch^{k-j} |f|_{k,p}$$

for $j = 0, 1, \ldots, k$ where C is a constant independent of f, h, and Z.

COROLLARY 2. *Suppose f satisfies the hypothesis of Corollary* 1. *Then f is in $W^{k,p}(R^n)$ and*

$$C_1 \|f\|_{k,p} \leqslant (1+h^k) |f|_{k,p} \leqslant C_2(1+h^k) \|f\|_{k,p}$$

where C_1 and C_2 are positive constants independent of f, h, and Z.

3. APPLICATION

Recall that the space $H_0^k(\Omega)$, where Ω is an open subset of R^n, is the closure in the $W^{k,2}(\Omega)$ norm of the class of infinitely differentiable functions with compact support in Ω. Let $B(u, v)$, which is defined by

$$B(u, v) = \int_\Omega \sum c_{\alpha,\beta} D^\alpha u \overline{D^\beta v}\, dx$$

where the c's are measurable functions and the sum is taken over $0 \leqslant |\alpha|$, $|\beta| \leqslant k$, be a Hermitian bilinear form on $H_0^k(\Omega)$ so that $(B(u, u))^{1/2}$ defines a semi-norm equivalent to $|u|_{k,2,\Omega}$. Consider the following interpolation problem:

Given a finite subset $G = \{x_1, ..., x_m\}$ of Ω and numbers $d_1, ..., d_m$, find u in $H_0^k(\Omega)$ such that $B(u, u) = \min\{B(v, v)$: $v \in H_0^k(\Omega)$ and such that $v(x_i) = d_i$, $x_i \in G$, $i = 1, ..., m\}$. (9)

In the case $n = 1$, $\Omega = [a, b]$, and $B(u, v) = \int_a^b D^2u \overline{D^2v}\, dt$, the solution of (9) is the well known cubic spline with zero boundary conditions (see [2]). Solutions of (9) in the general case, if they exist, may also be regarded as splines. The general problem has a unique solution under some very mild conditions on B and Ω (see [6]). Related problems have been considered in [5, 8, 11]; for a survey of various methods of multivariate interpolation see [6, 12].

Getting back to problem (9), take any g in $H_0^k(\Omega)$ such that $g(x_i) = d_i$, $x_i \in G$, $i = 1, ..., m$. If u is a solution of (9), we are interested in an estimate of $\|u - g\|_{L^2(\Omega)}$ in terms of g and G. The estimate below results from an application of Corollary 1. Such an estimate in the case when G is a rectangular lattice was given in [6]; indeed, the theorem below is a simple generalization of a portion of [6, Theorem 2.9]. The technique employed here can be used to obtain estimates for the related problems mentioned above; we intend to say more about this elsewhere. Concerning estimates for other methods of interpolation, it appears that they must be studied case by case.

THEOREM 2. *Suppose* $2k > n$, *u is a solution of* (9), *and g is any element in* $H_0^k(\Omega)$ *such that* $g(x_i) = d_i$, $x_i \in G$, $i = 1,..., m$. *If* $h = \max\{\text{distance}(x, G \cup \partial\Omega) : x \in \Omega\}$ *then*

$$|g - u|_{j,2,\Omega} \leqslant Ch^{k-j} \, |g|_{k,2,\Omega}$$

for $j = 0,..., k-1$ *and C is a constant independent of u, g, h, and G.*

Proof. Let V be the linear variety of $H_0^k(\Omega)$ consisting of those v's for which $v(x_i) = d_i$, $x_i \in G$, $i = 1,..., n$. Since $\{B(v, v)\}^{1/2}$ defines a Hilbert space norm equivalent to $\|v\|_{k,2,\Omega}$ on $H_0^k(\Omega)$, it follows from the elementary theory of Hilbert space that if v is in V then $v - u$ is perpendicular to u with respect to the corresponding inner product, namely, $B(v-u, u) = 0$. In particular $B(g, g) = B(g-u, g-u) + B(u, u)$ and thus $B(g-u, g-u) \leqslant B(g, g)$. Since $g(x_i) - u(x_i) = 0$, $x_i \in G$, $i = 1,..., m$, extending $g - u$ to be zero outside of Ω, applying Corollary 1 with $f = g - u$, and using the fact that $\{B(u, u)\}^{1/2}$ is a norm equivalent to $|u|_{k,2}$ we have, for $j = 0, 1,..., k-1$,

$$\begin{aligned} |g-u|_{j,2,\Omega} = |g-u|_{j,2} &\leqslant C_1 h^{k-j} \, |g-u|_{k,2} \leqslant C_2 h^{k-j} \{B(g-u, g-u)\}^{1/2} \\ &\leqslant C_2 h^{k-j} \{B(g, g)\}^{1/2} \leqslant C_3 h^{k-j} \, |g|_{k,2,\Omega}. \end{aligned}$$

Since this string of inequalities contains the desired result the proof is complete.

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